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Integral closure of invariant ideals, toroidal resolution, and equivariant vector bundles

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Abstract

We establish a connection between equivariant integrally closed ideal sheaves on a G-fibration Y over a G-spherical variety X with an affine fiber V and equivariant vector bundles on the universal toroidal resolution of X. As an application, we reduce the study of invariant integrally closed ideals of $V \times X$ to that of some smaller variety in the case of $X = M_{n,m}$. Moreover, we present an affirmative answer to a problem raised by Michel Brion [Comment. Math. Helv. 66 (1991) 237–262] for two special infinite series.

MSC: 13A15; 16W22

1. Introduction

One of the basic problem in the invariant theory is to describe the multiplicative structure of the coordinate ring $k[X]$ of an affine G-variety X via its G-module structure. After the pioneering work of De Concini et al. [8] in the case of $G = GL_n \times GL_m$ and $X = M_{n,m}$ (the space of $n \times m$ matrices), Ruitenburg [21] has derived such a result for an arbitrary prehomogeneous compactification of a semisimple symmetric space. Thus, it is natural to ask about the possibility to generalize it to the case of affine spherical varieties [4] or the

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multicone over complete symmetric varieties in the sense of [9,11], since both cases are multiplicity-free in a reasonable sense. At present, the former problem is still open while the latter problem was solved by Chirivì and Maffei [6]. In this paper, we present a method to deal with integrally closed ideals on the total space $X_{\mathscr{V}}$ of a vector bundle $\mathscr{V}$ on a spherical variety X (Theorem 3.13). Our framework contains both the scope of that of Brion and Faltings. However, we cannot solve their problems in general since our knowledge about vector bundles is not enough at present. In the case of $G = GL_n \times GL_m$ and $X_V = V \oplus M_{n,m}$, we can apply our previous work [14] to describe the structure of the integrally closed ideals on the total space (Theorem 4.6). Using this, we verify Brion's problem in the corresponding cases (Corollary 4.16). As far as the author knows, these seem to be the first cases that the generic stabilizer is not reductive.

Part of the results of this paper was announced (in Japanese) in [13].

Throughout this paper, k denotes some algebraically closed field of characteristic zero, G denotes a reductive algebraic group over k, and X denotes a normal G-algebraic variety over k.

2. Preliminary materials

Most of the materials in this section are standard. We include this section for the convenience of readers.

Definition 2.1 (*cf. Eisenbud [10, Section 4]*). An ideal sheaf $\mathscr{I}$ of $\mathscr{O}_X$ is called an integrally closed ideal sheaf if and only if there exists an affine chart $\{\mathrm{Spec} A_\lambda\}_{\lambda \in \Lambda}$ of X such that each $\Gamma(\mathrm{Spec} A_\lambda, \mathscr{I})$ is an integrally closed ideal of A_λ.

The next theorem is our starting point.

Theorem 2.2 (*Lipman [19, Sections 5–6]*). *Let $\mathscr{I}$ be a coherent ideal sheaf of X. Then, the following three conditions are equivalent*:

1. *$\mathscr{I}$ is integrally closed.*
2. *For all varieties X' and all birational proper morphisms $\pi : X' \to X$, we have $\mathscr{I} = \pi_*(\mathscr{I}\mathscr{O}_{X'})$.*
3. *There exists a normal variety X' and a birational proper morphism π such that $\mathscr{I}\mathscr{O}_{X'}$ is invertible and $\mathscr{I} = \pi_*(\mathscr{I}\mathscr{O}_{X'})$.*

Definition 2.3 (*cf. Knop [16]*). A normal G-variety X is called spherical if and only if X has a open dense orbit with respect to the action of a Borel subgroup $B \subset G$.

Theorem 2.4 (*Local structure theorem of Brion et al. [5]*). *Let $z \in X$ be a point in a closed G-orbit of X. Then, there exists a parabolic subgroup $P \subset G$, its Levi decomposition $P = LU$, and a locally closed affine subset $Z \subset X$ such that*:

1. *We have $z \in Z$;*
2. *L acts on Z and makes Z into a L-spherical variety*;

3. *There exists an open embedding* $P \times_L Z \hookrightarrow X$;
4. *We have* $P = \{g \in G; g(P \times_L Z) = P \times_L Z \subset X\}$.

Definition 2.5 (*cf.* Knop [16, Section 5]).

- *toroidal spherical variety*. Under the same setting as in Theorem 2.4, we call X a toroidal spherical variety if and only if $[L, L]$ acts on Z trivially for every choice of z. Equivalently, X is toroidal if and only if Z is a toric variety of some quotient torus T_0 of L (or a quotient torus of a maximal torus T of L) for every choice of z.
- *toroidal resolution of a spherical variety*. A G-equivariant birational proper map $\hat{X} \to X$ of G-spherical varieties is called a toroidal resolution if and only if $\hat{X}$ is a toroidal G-spherical variety.

Theorem 2.6 (*cf.* Brion [2, Section 1.4]). *Assume that X is a toroidal G-spherical variety. Let $\mathcal{O} \subset X$ be the unique open B-orbit in X. Then we have $P \cong \{g \in G; g\mathcal{O} = \mathcal{O}\}$ in Theorem* 2.4. *Here we have an equality if $B \subset P$. Moreover, X is smooth if and only if Z is smooth.*

Corollary 2.7. *Under the same assumptions as in Theorem* 2.2, *we further assume that X is G-spherical and $\mathscr{I}$ is a G-invariant subsheaf of $\mathcal{O}_X$. Then, we can choose $\pi : \hat{X} \to X$ to be a toroidal resolution.*

Proof. This is immediate from Theorem 2.2(1) $\Leftrightarrow$ (2) since every spherical variety admits a toroidal resolution. $\square$

Corollary 2.8. *Let X^+ be a smooth G-variety equipped with a G-equivariant birational proper map $\pi : X^+ \to X$. Then, for each G-equivariant* (*coherent*) *line subbundle $\mathscr{L}$ of $\mathcal{O}_{X^+}$, $\pi_* \mathscr{L}$ is an integrally closed G-equivariant ideal sheaf of $\mathcal{O}_X$.*

Proof. By considering at the level of local rings, a line subbundle $\mathscr{L} \subset \mathcal{O}_{X^+}$ is an integrally closed ideal sheaf. Hence, the statement follows from Theorem 2.2(3). $\square$

3. Integrally closed ideals and vector bundles

In this section, we assume that X is G-spherical.

3.1. Standard ideal sheaves

Let $\mathscr{V}$ be a G-equivariant vector bundle on X. We put $S^n_X(\mathscr{V}^\vee)$ as the n-th symmetric power of the $\mathcal{O}_X$-dual of $\mathscr{V}$. Consider its total space $X_\mathscr{V} := \mathrm{Spec}_X \bigoplus_{n \geqslant 0} S^n_X(\mathscr{V}^\vee)$. We denote the projection $X_\mathscr{V} \to X$ by $p_\mathscr{V}$. In this setting, we consider $X_\mathscr{V}$ as a $\mathbb{G}_m \times G$-variety as follows:

$$(\mathbb{G}_m \times G) \times V \times \mathscr{U} \ni (s, g) \times (v, x) \mapsto (sgv, gx) \in V \times g.\mathscr{U}$$

for every local trivialization $V \times \mathscr{U}$ of $X_\mathscr{V}$.

Remark 3.1. When X is affine, the equivariant Serre theorem (e.g. see [17, 1.1]) asserts that there exists a G-module V such that $X_{\mathscr{V}} \cong V \times X$ as an algebraic variety.

For the sake of simplicity, we put $H := \mathbb{G}_m \times G$. Then, a H-equivariant ideal sheaf $\mathscr{I}$ of $X_{\mathscr{V}}$ has the following description:

$$\mathscr{I} = \bigoplus_{n \geqslant 0} \mathscr{I}(n) \subset \bigoplus_{n \geqslant 0} S_X^n(\mathscr{V}^\vee).$$

The graded structure on the RHS comes from the first factor of H (the scalar multiplication along V). Notice that each $\mathscr{I}(n)$ is a torsion-free submodule of $\mathscr{S}_X^n(\mathscr{V}^\vee)$ since $\mathscr{O}_X$ is torsion-free. From now on, we assume that $\mathscr{I}$ is integrally closed. We apply Theorem 2.2 to the case $X_{\mathscr{V}}$ and $\mathscr{I}$. Then, we obtain a normal variety X^+ and a line bundle $\mathscr{L}$ associated to $\mathscr{I}$. Let X^{++} be the normalized blow-up along (pullbacks of) G-equivariant ideal sheaves on X which yields a toroidal resolution $\hat{X} \xrightarrow{\hat{\pi}} X$ (see [4, Theorem 3.3]). By abuse of notation, we write $\hat{X}_{\mathscr{V}}$ instead of $\hat{X}_{\hat{\pi}^*\mathscr{V}}$. By Theorem 2.2(3), we have a natural H-equivariant factorization morphisms

$$X^+ \xleftarrow{\pi^+} X^{++} \xrightarrow{\pi^-} \hat{X}_{\mathscr{V}} \xrightarrow{\hat{\pi}} X_{\mathscr{V}}.$$

Since all of the above H-varieties are normal and the morphisms are dominant birational, we have

$$\pi_*^-(\pi^+)^*\mathscr{O}_{X^+} \cong \mathscr{O}_{\hat{X}_{\mathscr{V}}} \quad \text{and} \quad \hat{\pi}_*\mathscr{O}_{\hat{X}_{\mathscr{V}}} \cong \mathscr{O}_{X_{\mathscr{V}}}.$$

We put $\mathscr{I}_t := \pi_*^-(\pi^+)^*\mathscr{L}$. It is clearly a H-equivariant integrally closed ideal sheaf of $\hat{X}_{\mathscr{V}}$. By assumption, we have $\hat{\pi}_*(\mathscr{I}_t) \cong \mathscr{I}$. Hence, we can deal with $\mathscr{I}_t$ instead of $\mathscr{L}$ or $\mathscr{I}$. As a $\mathscr{O}_{\hat{X}}$-module, we have the following $\mathbb{G}_m$-isotypical decomposition:

$$\bigoplus_{n \geqslant 0} \mathscr{I}_t(n) \subset \bigoplus_{n \geqslant 0} S_{\hat{X}}^n(\mathscr{V}^\vee) = \mathscr{O}_{\hat{X}_{\mathscr{V}}}.$$

Hence, $\mathscr{I}_t(n)$ is a G-equivariant coherent subsheaf of $S_{\hat{X}}^n(\mathscr{V}^\vee)$ for each n. Conversely, let $\{\mathscr{J}(n)\}_{n \geqslant 0}$ be a collection of G-equivariant quasi-coherent subsheaves of $\{S_{\hat{X}}^n(\mathscr{V}^\vee)\}_{n \geqslant 0}$. Consider their direct sum $\mathscr{J} := \bigoplus_{n \geqslant 0} \mathscr{J}(n) \subset \bigoplus_{n \geqslant 0} S_{\hat{X}}^n(\mathscr{V}^\vee) = \mathscr{O}_{\hat{X}_{\mathscr{V}}}$. Then, $\mathscr{J}$ is a H-equivariant ideal sheaf of $\mathscr{O}_{V \times \hat{X}}$ if and only if the image of the composition map

$$S_{\hat{X}}^n(\mathscr{V}^\vee) \otimes \mathscr{J}(m) \hookrightarrow S_{\hat{X}}^n(\mathscr{V}^\vee) \otimes S_{\hat{X}}^m(\mathscr{V}^\vee) \to \mathscr{S}_{\hat{X}}^{n+m}(\mathscr{V}^\vee)$$

is contained in $\mathscr{J}(n+m)$ for each $n, m \in \mathbb{Z}_{\geqslant 0}$. For a G-equivariant coherent subsheaf $\mathscr{W} \subset S^n(\mathscr{V}^\vee)$, we define $\mathscr{W}^m$ as the image of $\mathscr{W}^{\boxtimes m} \subset S^n(\mathscr{V}^\vee)^{\boxtimes m}$ under the map

$$S^n(\mathscr{V}^\vee)^{\boxtimes m} \longrightarrow S^{nm}(\mathscr{V}^\vee).$$

3.2. Reduction to the toric case

In this subsection, we work on $\hat{X}$, a toroidal resolution of a G-spherical variety X. In particular, we can apply our reduction technique to $\mathscr{I}_t$ defined in the previous subsection.

Corollary 3.2. *We assume the same assumptions* (*and notations*) *as in Definition* 2.5. *Let* $\mathscr{V}$ *be a G-equivariant vector bundle on X. We put* $Z_{\mathscr{V}} := p_{\mathscr{V}}^{-1}(Z)$. *Then, a H-equivariant ideal sheaf* $\mathscr{I}$ *on* $\hat{X}_{\mathscr{V}}$ *is integrally closed if and only if* $\mathscr{I}|_{Z_{\mathscr{V}}}$ *is integrally closed.*

Proof. Put $\mathscr{U} := (P \times_L Z) \subset \hat{X}$. Consider the restriction to $\mathscr{U}_{\mathscr{V}} := p^{-1}(\mathscr{U})$. Integral closure commutes with localization. Hence, if $\mathscr{I}$ is integrally closed, then $\mathscr{I}|_{\mathscr{U}_{\mathscr{V}}}$ is integrally closed. The unipotent radical U of P is isomorphic to an affine plane. Since $\mathscr{I}|_{\mathscr{U}_{\mathscr{V}}}$ is P-equivariant, it is of the form $\mathscr{I}|_{Z_{\mathscr{V}}} \otimes_k k[U]$. Therefore, $\mathscr{I}|_{\mathscr{U}_{\mathscr{V}}}$ is integrally closed if and only if $\mathscr{I}|_{Z_{\mathscr{V}}}$ is integrally closed. Thus, the assertion $\Rightarrow$ is proved. To prove the converse, it remains to show that $\mathscr{I}$ is integrally closed if $\mathscr{I}|_{\mathscr{U}_{\mathscr{V}}}$ is so. Since we know the G-equivariance of $\mathscr{I}$ a priori, $\mathscr{I}|_{g\mathscr{U}_{\mathscr{V}}}$ is integrally closed for every $g \in G$. Since $G.Z_{\mathscr{V}} = G.\mathscr{U}_{\mathscr{V}} = \hat{X}_{\mathscr{V}}$, gluing them yields the result. $\square$

Therefore we can check integrally closedness via restriction to a vector bundle over a toric variety Z (coming from Definition 2.5).

Another technique coming from the theory of toric varieties is the following.

Theorem 3.3 (*Klyachko [15, Section 2.2.1, Proposition 1]*). *Let* T_0 *be a split torus over k. Let* Z_0 *be a smooth affine* T_0*-toric variety. Then, for every* T_0*-equivariant vector bundle* $\mathscr{E}$ *on* Z_0, *there exists a* T_0*-module E such that*

$$\mathscr{E} \cong E \times Z_0$$

as T_0*-equivariant vector bundles.*

For a torus S, we define

$$X^*(S) := \mathrm{Hom}(S, \mathbb{G}_m) \quad \text{and} \quad X_*(S) := \mathrm{Hom}(\mathbb{G}_m, S).$$

Corollary 3.4. *In the same settings as in Theorem* 3.3, *there exist* $\chi_1, \ldots, \chi_e \in X^*(T_0)$ *such that*:

$$\mathscr{E} \cong E \times Z_0 \cong \bigoplus_{i=1}^{e} \chi_i \times Z_0.$$

Here we put $e := \dim E = \mathrm{rk}\mathscr{E}$. *Moreover, if we have* $\chi_i k[Z_0] \subset k[Z_0]$ *for every i, then we have a trivial* T_0*-module* E_0 *together with its k-basis* $v_1 \ldots v_e$ *such that*:

$$\mathscr{E} \cong E \times Z_0 \cong \bigoplus_{i=1}^{e} (\chi_i \otimes_k v_i) \times Z_0 \subset E_0 \times Z_0.$$

Corollary 3.5. *In the same settings as in Theorem* 3.3, *let E be a trivial T_0-module. Let $\mathscr{F}$ be a T_0-equivariant coherent subsheaf of $W \otimes_k \mathscr{O}_{Z_0}$. Then, there exist $w_1, w_2, \dots, w_p \in E$ and $\chi_1, \chi_2, \dots, \chi_p \in X^*(T_0)$ such that*

$$\mathscr{F} = \sum_{i=1}^{p} w_i \otimes_k \chi_i \otimes_k \mathscr{O}_{Z_0} \subset E \otimes_k \mathscr{O}_{Z_0}.$$

Proof. We can deal with $F \otimes k[Z_0] := \Gamma(Z_0, \mathscr{F})$ instead of $\mathscr{F}$ since Z_0 is affine. Then, $F \otimes k[Z_0]$ is a T_0-invariant finitely generated $k[Z_0]$-module. Choose T_0-invariant generators $m_1, \dots, m_p$ of F as a $k[Z_0]$-module. Let χ_i be a T_0-character of m_i for each $i = 1, \dots, p$. Then, we have

$$\chi_i^{-1} m_i \subset E \otimes_k k \subset E \otimes_k k[Z_0]$$

for each i. Hence, there exists $w_i \in E$ such that $m_i k[Z_0] = w_i \otimes_k \chi_i k[Z_0]$ as a submodule of $E \otimes_k k[Z_0]$ for each i. Therefore, the result follows. □

The following is a corollary of Klyachko's description of the category of torus-equivariant vector bundles on a toric variety (in terms of filtrations).

Corollary 3.6 (*cf. Klyachko [15]*). *Let T_0 be a split torus over k. Let Z be a smooth T_0-toric variety. A T_0-equivariant vector bundle on a union of open affine toric subvarieties $Z_1, Z_2, \dots$ is a restriction of some T_0-equivariant vector bundle on Z if there exists no open affine toric subvariety $Z^\sharp \subset Z$ which intersects two of Z_i's nontrivially*. (*Here intersection of toric subvarieties is called trivial if and only if the intersection is the open torus orbit.*)

Let Σ be the fan corresponding to Z and let $\Sigma_1, \Sigma_2, \dots$ be the fans corresponding to $Z_1, Z_2, \dots$. Then, the above condition is equivalent to the following: *there exists no cone $\sigma \in \Sigma$ such that*

$$\operatorname{supp}\sigma \cap \operatorname{supp}\Sigma_i \neq \{0\} \neq \operatorname{supp}\sigma \cap \operatorname{supp}\Sigma_j \quad for\ some\ i \neq j.$$

Lemma 3.7. *Let $\mathscr{W}$ be a G-equivariant subcoherent sheaf of $\mathscr{V}^\vee$. If $\mathscr{W}$ is a G-equivariant vector bundle, then $\bigoplus_{n \geqslant 0} \mathscr{W}^n \subset \bigoplus_{n \geqslant 0} S^n_{\hat{X}}(\mathscr{V}^\vee)$ is integrally closed in $\mathscr{O}_{\hat{X}_{\mathscr{V}}}$.*

Proof. We use the same notations as in Definition 2.5. We consider the restriction $\mathscr{W}|_Z$. We again localize to assume that Z is affine. By the description of Theorem 3.3, there exists a T-module W and V such that $\Gamma(Z, \mathscr{W}|_Z) \cong W \otimes k[Z] \subset V \otimes k[Z]$. Hence, we have an inclusion

$$\bigoplus_{n \geqslant 0} \Gamma(Z, \mathscr{W}^n|_Z) \cong \bigoplus_{n \geqslant 0} S^n(W) \otimes_k k[Z] \subset k[V] \otimes_k k[Z] \cong k[Z_{\mathscr{V}}].$$

Since the quotient fields of the both sides are the same, we obtain the result. □

3.3. Sectionally closed sheaves

Let $\hat{X}$ be a smooth toroidal resolution of X. In this subsection, we introduce a variant of integrally closedness, which is weaker than the original.

Definition 3.8 (*Sectionally closed sheaves*).

- *Toric case*. Let T_0 be a (split) torus over k. Let Y be an affine smooth T_0-toric variety. Let V be a T_0-module. Then, a T_0-equivariant coherent subsheaf $\mathscr{W}_0$ of $V\otimes_k\mathscr{O}_Y$ is called sectionally closed if and only if for each $v \in V$, the intersection

$$\mathscr{W}_0 \cap (kv\otimes_k\mathscr{O}_Y) \subset \mathscr{O}_Y \overset{v}{\hookrightarrow} V\otimes_k\mathscr{O}_Y$$

is an integrally closed ideal sheaf of $\mathscr{O}_Y$.
- *Spherical case*. Assume that $\mathscr{W}$ is a G-equivariant vector bundle on $\hat{X}$. We call a G-equivariant coherent subsheaf $\mathscr{V}$ of $\mathscr{W}$ sectionally closed if and only if it is sectionally closed when restricted to every affine toric subvariety $Z_0 \subset Z$ coming from Definition 2.5. (cf. Theorem 3.3)

Lemma 3.9 (*Key observation*). *Let $\mathscr{I}$ be a H-equivariant integrally closed ideal sheaf of $\hat{X}_{\mathscr{V}}$. Then, each degree-component $\mathscr{I}(n) \subset S^n_{\hat{X}}(\mathscr{V}^\vee)$ is sectionally closed. Let Z be as in Definition* 2.5. *Let $Z_0 \subset Z$ be an affine T-open subset. Let V_n be a T-module such that there exists an inclusion $S^n_{Z_0}(\mathscr{V}^\vee) \subset V_n\otimes_k\mathscr{O}_{Z_0}$ of T-equivariant coherent sheaves which is an isomorphism on the dense open T-orbit $T_0 \subset Z_0$. Then $\mathscr{I}(n)|_{Z_0} \subset V_n\otimes_k\mathscr{O}_{Z_0}$ is also sectionally closed.*

Proof. By Corollary 3.2, $I := \Gamma(p^{-1}_{\mathscr{V}}(Z_0), \mathscr{I}|_{p^{-1}(Z_0)}) \subset k[p^{-1}(Z_0)]$ is integrally closed. For each $v \in S^n_{T_0}(\mathscr{V}^\vee)$, consider the subring

$$\mathscr{A} := \bigoplus_{m\geqslant 0} kv^{\otimes m}\otimes_k\mathscr{O}_{Z_0} \subset \bigoplus_{m\geqslant 0} \mathscr{S}^{nm}_{T_0}(\mathscr{V}^\vee) \subset \mathscr{O}_{V_n\times T_0}.$$

We put $A := \Gamma(Z_0, \mathscr{A})$. It is clear that A is integrally closed. We have $Q(A) \subset Q(k[p^{-1}(Z_0)])$. Hence, $A \cap I$ is an integrally closed ideal of A. Thus, $kv\otimes_k\mathscr{O}_{Z_0} \cap \mathscr{I}_{Z_0}$ is also integrally closed as an ideal sheaf of $\mathscr{O}_{Z_0}$. Therefore, the second assertion follows. The first assertion is obtained by glueing Z_0's. $\square$

The main theorem of this subsection is the following.

Theorem 3.10. *Let n be an integer. Let $\mathscr{I}$ be a H-equivariant integrally closed ideal sheaf on $\hat{X}_{\mathscr{V}}$. Then, there exists a toroidal G-spherical variety $\hat{X}^+$ and a G-equivariant coherent subsheaf $\mathscr{I}^+(n)$ of $S^n_{\hat{X}^+}(\mathscr{V}^\vee)$ such that*:

- *there exists a G-equivariant dominant morphism $\psi : \hat{X}^+ \to \hat{X}$ obtained by a successive G-equivariant normalized blow-ups*;

- *$\mathscr{I}^+(n)$ is a (G-equivariant) vector bundle on $\hat{X}^+$;*
- *$\psi_* \mathscr{I}^+(n) = \mathscr{I}(n)$ as a subcoherent sheaf in $S^n_{\hat{X}}(\mathscr{V}^\vee)$.*

Proof. We adopt the same notations as in Definition 2.5. We consider the restriction $\mathscr{I}(n)|_Z$. We again localize and work on an affine T-subvariety $Z_0 \subset Z$. Let $T^0 := \ker[T \to T_0]$. We fix a section $X^*\left(T^0\right) \subset X^*(T)$. Consider a T-module W such that:

- we have $\operatorname{rk} S^n_{\hat{X}}(\mathscr{V}^\vee) = \dim W$;
- there exist a T_0-character ρ and trivial T_0-modules W_χ such that
$$W \otimes_k \rho = \bigoplus_{\chi \in X^*(T^0)} \chi \otimes_k W_\chi;$$
- we have $S^n_{Z_0}(\mathscr{V}^\vee) \subset W \otimes_k \mathscr{O}_{Z_0}$ as T-equivariant coherent sheaves.

Then, $\mathscr{I}(n)|_{Z_0}$ is a sectionally closed coherent subsheaf of $W \otimes_k \mathscr{O}_{Z_0}$ by Lemma 3.9. For each $w \in W$, consider the intersection $I_w := (kw \otimes_k k[Z_0]) \cap \mathscr{I}(n)|_{Z_0}$ in $W \otimes_k k[Z_0]$. I_w is an integrally closed ideal of $k[Z_0]$. Moreover, we have $\mathscr{I}(n)|_{Z_0} = \sum_{w \in W} I_w$ by Corollary 3.5. Since $\mathscr{I}(n)$ is coherent, it is finitely generated (say the number of generators m). There are finitely many types of $\{I_w\}_{w \in W}$ (as ideals of $k[Z_0]$), which is majorated by 2^m. Consider a successive normalized blow-ups Z_0^+ of Z_0 along all of I_w's, regarded as (toric) ideals of $k[Z_0]$. We denote the natural dominant map $Z_0^+ \to Z_0$ by ϕ. Then, for each $w \in W$, there exists a minimal line subbundle $I_w^+ \subset W \otimes_k \mathscr{O}_{Z_0^+}$ on Z_0^+ such that $\phi_* I_w^+ = I_w$. Take a linear hull $\mathscr{I}(n)_{Z_0^+}$ of $\{I_w^+\}_{w \in W}$ in$W \otimes_k \mathscr{O}_{Z_0^+}$. We have $\mathscr{I}(n) \subset \phi_* \mathscr{I}(n)_{Z_0^+}$. By construction, we have $\phi^{-1}\mathscr{I}(n) \cdot \mathscr{O}_{Z_0^+} = \mathscr{I}(n)_{Z_0^+}$. Therefore, $\mathscr{I}(n)_{Z_0^+} \subset W \otimes_k \mathscr{O}_{Z_0^+}$ has a property that $kw \otimes_k \mathscr{O}_{Z_0^+} \cap \mathscr{I}(n)_{Z_0^+}$ is a line bundle for every $w \in W$. Here we need the following lemma:

Lemma 3.11. *There exists a T-equivariant morphism $\theta : \tilde{Z}_0^+ \to Z_0^+$ and a T-equivariant vector bundle $\tilde{\mathscr{I}}(n)$ on $\tilde{Z}_0^+$ such that $\theta_* \tilde{\mathscr{I}}(n) \cong \mathscr{I}(n)_{Z_0^+}$.*

Proof of Lemma 3.11. Choose an (arbitrary) affine open toric subvariety $Z^- \subset Z_0^+$. We introduce a coordinate system $z_1, \ldots, z_r$ such that $k[Z^-] = k[z_1, z_2, \ldots, z_r]$. For each $1 \leqslant i \leqslant r$, we put $k[Z^-]_i := k[z_1^{\pm 1}, \ldots, z_{i-1}^{\pm 1}, z_i, z_{i+1}^{\pm 1}, \ldots]$.

Claim 1. *We have*:

($\bullet$) *Let $\mathscr{J} \subset W \otimes k[Z^-]$ be such that $kw \otimes k[Z^-] \cap \mathscr{J}$ is a free $k[Z^-]$-module of rank one for every $w \in W$. Then there exist a family of T-modules $\{J_i\}_{i=1}^r$ such that*
$$\mathscr{J} = \bigcap_{i=1}^r J_i \otimes k[Z^-]_i.$$

Proof of Claim 1. By Theorem 3.3, we have $\dim W$-dimensional T-modules W_i such that $W \otimes k[Z^-] \subset W_i \otimes k[Z^-]$ and $\mathscr{J} \subset \bigcap_{i=1}^r W_i \otimes k[Z^-]_i$. Here the inclusion is an isomorphism up to codimension two locus in Z^-. Since $\mathscr{J} \cap (kw \otimes k[Z^-])$ is a line bundle for every $w \in W$, we conclude

$$\mathscr{J} = \bigcap_{i=1}^{r} W_i \otimes k[Z^-]_i$$

as desired. $\square$

Let $\sigma = \sum_{i=1}^{r} \mathbb{R}_{\geqslant 0} e_i \subset X_*(T_0) \otimes_{\mathbb{Z}} \mathbb{R}$ be the fan corresponding to $k[Z^-]$ (i.e. $\langle e_i, z_j \rangle = \delta_{i,j}$). Consider a fan Σ such that (1) Σ is a subdivision of σ in the sense of Oda [20] and (2) Σ contains cones

$$\sigma_i = \mathbb{R}_{\geqslant 0} e_i + \sum_{j \neq i} \mathbb{R}_{\geqslant 0}(ce_i + e_j) \quad (1 \leqslant i \leqslant r),$$

where c is a sufficiently large integer. We define $\tilde{Z}^-$ as the T_0-toric variety corresponding to the fan Σ. We have a map $\theta : \tilde{Z}^- \to Z^-$. For each $\tau \in \Sigma$, we define $\tilde{Z}^-(\tau) \subset \tilde{Z}^-$. By Claim 1, we have a decomposition

$$\mathscr{I}(n)|_{Z^-} = \bigcap_{i=1}^{r} J_i \otimes k[Z^-]_i.$$

We put $\tilde{\mathscr{I}}(\sigma_i) := J_i \times \mathscr{O}_{\tilde{Z}^-(\sigma_i)}$ for each i. By Corollary 3.6, we have a T-equivariant vector bundle $\tilde{\mathscr{I}}(\Sigma)$ on $\tilde{Z}^-$ which is an extension of $\tilde{\mathscr{I}}(\sigma_i)$'s. Then, gluing $\tilde{Z}^-$'s yields $\tilde{Z}_0^+$ and glueing the resulting vector bundles on $\tilde{Z}^-$'s yields $\tilde{\mathscr{I}}(n)$. $\square$

We return to the proof of Theorem 3.10. Remember that $\mathscr{I}(n)_{Z_0^+}$ is a torus-equivariant torsion free sheaf. Hence, $\phi_* \mathscr{I}(n)_{Z_0^+}$, regarded as a submodule of $W \otimes_k k[Z_0]$, is decomposed into a sum of T-modules. Let ξ be a one-dimensional T-submodule of $\phi_* \mathscr{I}(n)_{Z_0^+}$. Let $\chi \in X^* \left(T^0\right)$ be a character such that $\xi \chi^{-1} \in X^* (T_0)$. (Here we regard $\chi \in X^* (T)$ via the fixed section.) Then, we have

$$k[T_0]\xi \cap \rho^{-1}\chi \otimes_k W_\chi \otimes_k k[Z_0] = k[T_0]\xi \cap W \otimes_k k[Z_0] \subset W \otimes_k k[T_0].$$

Hence, $\xi k[T_0] \cap W \otimes_k k[Z_0]$ written as $w \otimes_k k[Z_0]$ by some $w \in W$ by Corollary 3.4. Since each irreducible T-submodule of $\mathscr{I}(n)|_{Z_0}$ comes from some $\phi_* I_w^+ = I_w$, we conclude $\phi_* \mathscr{I}(n)_{Z_0^+} = \mathscr{I}(n)|_{Z_0}$. Finally, putting ψ to be the associated map $G.\tilde{Z}_0^+ \to G.Z_0$ and putting $\mathscr{I}(n)^+$ to be a G-equivariant vector bundle which corresponds to $\tilde{\mathscr{I}}(n)$ yields the result by gluing on toroidal G-spherical open covering. $\square$

Corollary 3.12. *Let $\mathscr{I}$ be a H-equivariant integrally closed ideal sheaf of $X_{\mathscr{V}}$. Then, each degree component gives rise to a G-equivariant vector bundle on some toroidal resolution $\hat{X}$ of X.*

Proof. Combine Lemma 3.9 and Theorem 3.10 degree-wise. $\square$

3.4. Reformulation via universal resolution

Thanks to Theorem 3.10, we can reduce the study of a degree component of an integrally closed ideal on $X_{\mathscr{V}}$ to the study of a vector bundle on a certain toroidal resolution $\hat{X}$ of X. However, we cannot tell how big $\hat{X}$ is. The aim of this subsection is to reformulate the above result by introducing the limit of toroidal resolutions. As a result, we can reduce the study of integrally closed ideals to the study of the asymptotic behavior of the category of equivariant vector bundles on toroidal resolutions of X as in the next section.

Let T_0 be a k-split torus. Let Z_0 be a T_0-toric variety. Let $\{\underline{\pi}^{\lambda} : Z_0^{\lambda} \to Z_0\}_{\lambda \in \Lambda}$ be a family of T_0-equivariant birational proper maps $Z_0^{\lambda} \to Z_0$ consisting of smooth toric varieties Z_0^{λ}. By putting $\underline{\pi}^{\lambda,\lambda'} : Z_0^{\lambda,\lambda'} \to Z_0$ to be the morphism from the T_0-equivariant desingularization of the normalization $Z_0^{\lambda,\lambda'}$ of $Z_0^{\lambda} \times_{Z_0} Z_0^{\lambda'}$, we can form the following T_0-equivariant commutative diagram:

$$\begin{tikzcd}
 & Z_0^{\lambda,\lambda'} \arrow[dl, "\underline{\pi}^{\lambda'}_{\lambda}"'] \arrow[dr, "\underline{\pi}^{\lambda}_{\lambda'}"] & \\
Z_0^{\lambda} \arrow[dr, "\underline{\pi}^{\lambda}"'] & & Z_0^{\lambda'} \arrow[dl, "\underline{\pi}^{\lambda'}"] \\
 & Z_0 &
\end{tikzcd}$$

such that $\underline{\pi}^{\lambda} \circ \underline{\pi}^{\lambda'}_{\lambda} = \underline{\pi}^{\lambda'} \circ \underline{\pi}^{\lambda}_{\lambda'} = \underline{\pi}^{\lambda,\lambda'}$. Hence, $\{\underline{\pi}^{\lambda} : Z^{\lambda} \to Z_0\}_{\lambda \in \Lambda}$ forms a net and we have a pro-object

$$Z_0^{\infty} := \varprojlim \{Z_0^{\lambda}\}_{\lambda \in \Lambda}$$

of T_0-toric variety which we call the universal toric variety over Z_0. Let X be a G-spherical variety and $\hat{X}$ be its minimal toroidal resolution (also called *décoloration* in [4]). From [16], we have the corresponding family of G-equivariant proper birational morphisms $\{\pi^{\lambda} : X^{\lambda} \to X\}_{\lambda \in \Lambda}$ which satisfies the following commutative diagram:

$$\begin{tikzcd}
P \times_L Z^{\lambda} \arrow[r, hook] \arrow[d, "1 \times \underline{\pi}^{\lambda}"'] \arrow[dr, phantom, "\circlearrowleft"] & X^{\lambda} \arrow[d] \arrow[dr, "\pi^{\lambda}"] & \\
P \times_L Z \arrow[r, hook] & \hat{X} \arrow[r] & X
\end{tikzcd}$$

Here T, Z are as in Definition 2.5 and $\{\underline{\pi}^{\lambda} : Z^{\lambda} \to Z\}$ is a projective system of T-smooth resolutions of Z. Therefore, we have a pro-object

$$X^{\infty} := \varprojlim \{X^{\lambda}\}_{\lambda \in \Lambda} \xrightarrow{\pi^{\infty}} X$$

of G-spherical varieties which we call the (universal) toroidal resolution of X.

For each $\mu \in \Lambda$, we define

$$\Lambda(\mu) := \{\lambda \in \Lambda; \exists \pi^{\lambda}_{\mu} : X^{\lambda} \to X^{\mu} : G\text{-equivariant birational proper map}\}.$$

Here $\pi^{\mu} \circ \pi^{\lambda}_{\mu} = \pi^{\lambda}$ automatically holds. Then, we define a vector bundle on X^{∞} as a G-equivariant locally free sheaf $\mathscr{E}$ on X^{μ} which we regard as a family $\{(\pi^{\lambda}_{\mu})^{*}\mathscr{E}\}_{\lambda \in \Lambda(\mu)}$. Here we define a morphism of vector bundles $\{(\pi^{\lambda}_{\mu})^{*}\mathscr{E}\}_{\lambda \in \Lambda(\mu)}$ and $\{(\pi^{\lambda}_{\gamma})^{*}\mathscr{F}\}_{\lambda \in \Lambda(\gamma)}$ as a G-equivariant morphisms of coherent sheaves $f_{\lambda} : (\pi^{\lambda}_{\mu})^{*}\mathscr{E} \to (\pi^{\lambda}_{\gamma})^{*}\mathscr{F}$ such that $(\pi^{\delta}_{\lambda})^{*} f_{\lambda} = f_{\delta}$ for every $\delta \in \Lambda(\lambda)$.

Then, Corollary 3.12 is re-expressed as follows:

Theorem 3.13. *Let $\mathscr{I}$ be a H-equivariant integrally closed ideal sheaf on $X_{\mathscr{V}}$. Then, there exists a H-equivariant integrally closed ideal sheaf $\mathscr{I}^{+}$ on $X^{\infty}_{\mathscr{V}}$ such that* (1) $\pi^{\infty}_{*}\mathscr{I}^{+} = \mathscr{I}$ *and* (2) $\mathscr{I}^{+}(n)$ *is a vector bundle on X^{∞} for every $n \in \mathbb{Z}_{\geqslant 0}$.*

4. Case $G = GL_n \times GL_m$ and $X = M_{n,m}$

We put $G = GL_n \times GL_m$ ($n \leqslant m$) and $X = M_{n,m}$ ($n \times m$ matrix space). We denote the diagonal torus of GL_n and GL_m by T_n and T_m, respectively. Let $\omega_1, \omega_2, \ldots, \omega_n$ be the weights of T_n defined as follows:

$$\omega_i : T_n \ni \begin{pmatrix} t_1 & 0 & \cdots & 0 \\ 0 & t_2 & \ddots & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & t_n \end{pmatrix} \mapsto t_i \in \mathbb{G}_m.$$

We also define $\alpha_1 := -\omega_1$ and $\alpha_i := -\omega_i + \omega_{i-1}$ for $i = 2, \ldots, n$. We put $R^{+} := \sum_{i=1}^{n} \mathbb{Z}_{\geqslant 0}\alpha_i \subset X^{*}(T_n)$. Let V be a GL_n-module. We consider the product variety $M_V := V \times M_{n,m}$. Let $M^{t}_{n,m}$ be the minimal toroidal resolution of $M_{n,m}$, i.e. the successive blow-up of $M_{n,m}$ with respect to the subsets $\overline{\{A \in M_{n,m}; \mathrm{rk} A \leqslant s\}}$ for $0 \leqslant s < n$. We set Z as in Definition 2.5. Then, we have $k[Z] = k[z_1, \ldots, z_n]$, where T_n-weight (with respect to the right T_n-action) of z_i is equal to α_i for $1 \leqslant i \leqslant n$. Let $M^{\infty}_{n,m}$, $M^{\infty}_{n,n}$, and Z^{∞} be the (universal) toroidal resolutions of $M_{n,m}$, $M_{n,n}$, and Z, respectively. From the natural $GL_n \times GL_n$-equivariant inclusion $M_{n,n} \hookrightarrow M_{n,m}$, we obtain a G-equivariant fibration

$$M^{\infty}_{n,m} \to GL_m/P_m,$$

with a fiber $M^{\infty}_{n,n}$. Here P_m is a parabolic subgroup of GL_m defined as follows:

$$P_m = \begin{pmatrix} GL_n & M_{n,m-n} \\ 0 & GL_{m-n} \end{pmatrix} \subset GL_m.$$

We denote the Levi decomposition of P_m by $P_m = L_m U_m$, where L_m is the Levi subgroup such that $T_m \subset L_m$ and U_m is the unipotent radical of P_m. The following lemma reduces the study of H-invariant integrally closed ideals on M_V to the case $n = m$. (Hence, we will assume $n = m$ in the proofs after Lemma 4.1.)

Lemma 4.1. *Let $\mathscr{I}$ be a H-equivariant integrally closed ideal sheaf $\mathscr{I}$ on M_V. We choose $\mathscr{I}^+$ as in Theorem* 3.13. *Then, $\mathscr{I}^+$ is determined uniquely by its restriction to $M^\infty_{n,n}$ (as a $GL_n \times GL_n$-equivariant ideal sheaf).*

Proof. Since U_m operates trivially on every fiber of M_V along $M_{n,m}$, U_m operates trivially on every fiber of M^∞_V along $M^\infty_{n,m}$. Here, every weight of $k[V \times Z^\infty] = k[V \times Z] \cong k[V] \otimes k[Z]$ is in the weight lattice of $T_n \times T_n$. Hence, we see that U operates trivially on $\mathscr{I}^+|_{Z^\infty} \subset k[V \times Z] \cong k[V] \otimes k[Z]$ by the comparison of $T_n \times T_m$-weights. It follows that U_m must operate on $\mathscr{I}^+|_{Z^\infty}$ trivially. Hence, the result follows by the standard material (cf. [7, 5.2.16]). □

4.1. Integrally closed ideals

Let

$$\mathfrak{gl}_n := \bigoplus_{\alpha \in \triangle} k e_\alpha \oplus \bigoplus_{i=1}^{n} k h_i$$

be the root space decomposition, where $\triangle \subset X^*(T_n)$ is the set of roots, e_α is a root vector corresponding to $\alpha \in \triangle$, and $(h_i)^n_{p,q=1} = \delta_{p,i}\delta_{q,i}$. We denote the Lie subalgebras of $\mathfrak{gl}_n$ generated by the upper (resp. lower) triangular matrices by $\mathfrak{b}$ (resp. $\mathfrak{b}^-$). Then, we define elements of $\mathfrak{gl}_n \otimes k[Z]$ as follows:

$$\tilde{e}_\alpha := \begin{cases} e_\alpha \otimes 1 & \text{if } e_\alpha \in \mathfrak{b}^- \\ e_\alpha \otimes z^\alpha & \text{if } e_\alpha \in \mathfrak{b} \end{cases}, \qquad \text{and} \quad \tilde{h}_i := h_i \otimes 1.$$

Here we set $z^\mu := \prod_{i=1}^n z_i^{n_i}$, where $\mu = \sum_{i=1}^n n_i \alpha_i \in X^*(T_n)$. We put

$$\mathfrak{p}^0 := \mathrm{LieSpan}\langle \tilde{e}_\alpha, \tilde{h}_i ; \alpha \in \triangle, i = 1, \ldots, n \rangle \subset \mathfrak{gl}_n \otimes k[Z].$$

For every $\mathfrak{gl}_n$-module V_0, $V_0 \otimes k[Z]$ is a $\mathfrak{gl}_n \otimes k[Z]$-module in a natural way. Thus, it is a $\mathfrak{p}^0$-module. Moreover, we define an action of T_n on $k[Z]$ from the right and prolong it to $V_0 \otimes k[Z]$ by letting T_n acts on V_0 trivially. Notice that these two actions do not commute each other.

Definition 4.2. We define a category $\mathscr{C}^V$ as follows:

Objects. An integrally closed ideal I of $k[V \times Z] \cong \bigoplus_{n \geqslant 0} S^n(V^*) \otimes k[Z]$ which is stable via $(\mathfrak{p}^0, T_n)$-action.

Morphisms. For each $I, J \in \mathbf{Ob}\mathscr{C}^V$, we define their morphism $f : I \to J$ as an inclusion of ideals in $k[V \times Z]$.

Definition 4.3. Let $\mathscr{C}^V_+$ be the category such that:

Objects. A H-equivariant integrally closed ideal sheaf $\mathscr{I}$ on M^∞_V such that $\mathscr{I}(n)$ is a vector bundle for every $n \in \mathbb{Z}_{\geqslant 0}$.

Morphisms. For each $\mathscr{I}, \mathscr{J} \in \mathbf{Ob}\mathscr{C}^V_+$, we define their morphism $f : \mathscr{I} \to \mathscr{J}$ as an inclusion of ideal sheaves on M^∞_V.

Let $\mathscr{C}_0^V$ be the category of H-invariant integrally closed ideals on M_V whose morphisms are inclusions as submodules of $k[V \times M_{n,n}]$ and let $\mathscr{C}_t^V$ be the category of H-equivariant integrally closed ideal sheaves on M_V^t whose morphisms are inclusions as submodules of $k[V] \otimes \mathcal{O}_{M_{n,n}^t}$.We define

$$\Sigma^\vee := \sum_{i=1}^{n} \mathbb{R}_{\geqslant 0}\alpha_i \subset X^*(T_n)\otimes_{\mathbb{Z}}\mathbb{R}.$$

We define Σ to be the dual cone of $\Sigma^\vee$ in $X_*(T_n)\otimes_{\mathbb{Z}}\mathbb{R}$. By abuse of notation, we also denote by Σ the fan consisting of a unique n-dimensional cone Σ and its faces.

For a $\mathfrak{gl}_n$-module V_0, we denote its T_n-weight set by $\mathrm{wt}\,V_0$. We define the diameter $\mathrm{diam}\,V_0$ of V_0 as follows:

$$\mathrm{diam}\,V_0 = \sum_{i=1}^{n} \max\{\lceil\langle\omega_i^\vee, \lambda - \mu\rangle\rceil; \lambda, \mu \in \mathrm{wt}\,V_0\}\alpha_i.$$

Here, $\omega_1^\vee, \ldots, \omega_n^\vee$ is the set of primitive generators of Σ defined as $\langle\omega_i^\vee, \alpha_j\rangle = \delta_{i,j}$.

Theorem 4.4 (*cf. Knop [16, Sections 3–4]*). *Let σ be a n-dimensional cone of $X_*(T_n)$ such that* $\mathrm{supp}\sigma \subset \mathrm{supp}\Sigma$. *Then, there exists a corresponding compactification $M(\sigma)$ of GL_n which is a* (*simple*) *smooth toroidal G-spherical variety with a morphism $\pi^\sigma : M(\sigma) \to M_{n,n}^t$.*

For a cone σ of $X_*(T_n)$, we define $k[\sigma^\vee]$ to be a subalgebra of $k[T_n]$ consisting of sum of monomials in $\sigma^\vee$. We have $k[Z] = k[\Sigma^\vee]$. The following is a special case of the main result of K [K04].

Theorem 4.5 (*Kato [14, Theorem 4.1]*). *Under the same settings as in Theorem* 4.4, *we further assume that $\sigma \cap \partial\Sigma = \emptyset$. The category of G-equivariant vector bundles on $X(\sigma)$ is equivalent to the category of the $\mathfrak{p}^0$-stable projective $k[\sigma^\vee]$-modules which are contained in $V_0 \otimes k[T_0]$ for some $\mathfrak{gl}_n$-module V_0.*

Theorem 4.6. *There exists a faithful dense functor $F : \mathscr{C}^V \to \mathscr{C}_0^V$.*

Proof. Let $\pi^c : M_{n,n}^\infty \to M_{n,n}^t$ be the natural surjective morphism. Then, the natural functor $\Gamma : \mathscr{C}_+^V \to \mathscr{C}_0^V$ factors through $\pi_*^c : \mathscr{C}_+^V \to \mathscr{C}_t^V$ and $\pi_*^t : \mathscr{C}_t^V \to \mathscr{C}_0^V$. For each $\mathscr{I} \in \mathbf{Ob}\mathscr{C}_0^V$, there exists a H-equivariant integrally closed ideal sheaf $\mathscr{I}_t$ such that $\pi_*^t\mathscr{I}_t = \mathscr{I}$. By Lemma 3.2, $\mathscr{I}_t|_{V\times Z}$ is also a T_n-invariant integrally closed ideal. Hence, if we prove that every $\mathfrak{p}^0$-stable T_n-invariant integrally closed ideal I_t on $V \times Z$ comes from $\mathscr{C}_+^V$ via direct image π_*^c and restriction $|_{V\times Z}$, then the result follows since the direct image of an integrally closed ideal sheaf is integrally closed. We have the following commutative diagram:

$$\begin{tikzcd}
P \times_L Z^\infty \arrow[r, hook] \arrow[d, "1\times\underline{\pi}^c"'] \arrow[dr, phantom, "\circlearrowleft"] & M^\infty_{n,n} \arrow[d, "\pi^c"] \arrow[dr] & \\
P \times_L Z \arrow[r, hook] & M^t_{n,n} \arrow[r, "\pi^t"] & M_{n,n}
\end{tikzcd}$$

Thus, all we have to prove is the surjectivity of $\underline{\pi}^c_*$. $I_t(p) \subset S^p_Z(V^*)$ is a T_n-equivariant sectionally closed submodule for every $p \in \mathbb{Z}_{\geqslant 0}$. Hence, there exists a subdivision Σ_1 of Σ such that (1) the corresponding T_n-toric variety Z_1 is smooth and (2) there exists a T_n-equivariant vector subbundle $\mathscr{J}(p) \subset S^p_{Z_1}(V^*)$ such that $\underline{\pi}^1_*\mathscr{J}(p) = I(p)$. Here $\underline{\pi}^1 : Z_1 \to Z$ is the T_n-equivariant morphism corresponding to the subdivision.

There exists a (possibly non-simplicial) subcone $\sigma_0 \subset \Sigma$ such that (1) $\sigma_0 \cap \partial\Sigma = \emptyset$ and (2) $k[\sigma_0^\vee]$ is equal to $k[Z]$ up to weight $2\mathrm{diam} S^p(V^*) = 2p\mathrm{diam} V^*$. Hence, there exists a subdivision (as a fan) Σ_2 of Σ_1 such that (1) σ_0 is a union of cones of Σ_2 and (2) the corresponding (partial) compactification $X(\Sigma_2)$ is smooth. We denote the corresponding T_n-toric variety by Z_2. We put the natural T_n-equivariant morphism $Z_2 \to Z$ by $\underline{\pi}^2$.

We denote an affine open subset of Z_2 corresponding to $\sigma \in \Sigma_2$ by Z_2^σ. Then, pullback of coherent sheaves via $\underline{\pi}^\sigma : Z_2^\sigma \to Z$ is expressed as the tensor product $\otimes_{k[Z_0]} k[\sigma^\vee]$. Hence, $(\underline{\pi}^\sigma)^* I_t(p)$ is $\mathfrak{p}^0$-stable.

We can write

$$(\underline{\pi}^\sigma)^* I_t(p) := \bigcap_{i=1}^{n} \left(\sum_{m \geqslant 0} F^i_\sigma(-m) \otimes_k \eta_i^m k[\sigma^\vee] \right) \subset S^p(V^*) \otimes k[\sigma^\vee]$$

by Klyachko's theorem (Theorem 3.3). Here $\eta_1, \ldots, \eta_n$ is the set of primitive generators of $\sigma^\vee$. We denote the dual generators of σ by $\eta_j^\vee$. (Thus, $\langle \eta_i^\vee, \eta_j \rangle = \delta_{i,j}$ holds.) Here we define $\tilde{F}^i_\sigma(m)$ by $U\left(\mathfrak{gl}_n\right) F^i_\sigma(m) \subset S^p(V^*)$ if $\langle \eta_i^\vee, \alpha_j \rangle = 0$ for some j and $F^i_\sigma(m)$ otherwise. Then, we define

$$\tilde{I}^\sigma_t(p) := \begin{cases} \bigcap_{i=1}^n \left(\sum_{m \geqslant 0} \tilde{F}^i_\sigma(-m) \otimes_k \eta_i^m k[\sigma^\vee] \right) & \text{if } \sigma \cap \partial\Sigma \neq \emptyset, \\ (\underline{\pi}^\sigma)^* I_t(p) & \text{if } \sigma \cap \partial\Sigma = \emptyset. \end{cases}$$

This gives rise to a $\mathfrak{p}^0$-stable T_n-equivariant vector bundle $\tilde{I}^\sigma_t(p)$ on Z_2. We have $(\underline{\pi}^2)^* I_t(p) \subset \tilde{I}^\sigma_t(p)$. Moreover, we can choose a common generator sets for $(\underline{\pi}^2)^* I_t(p)$ and $(\underline{\pi}^\sigma)^* I_t(p)$ since $k[Z]$ and $k[\sigma^\vee]$ coincides up to degree $2p\mathrm{diam} V^*$. (By the action of $\mathfrak{p}^0$, each filtration F^i_σ becomes stable after $\langle \eta_i^\vee, 2p\mathrm{diam} V^* \rangle$-step from the first nontrivial term on each irreducible component.) Therefore, we have $\underline{\pi}^2_* \tilde{I}^\sigma_t(p) = I(p)$. Thus, every object of $\mathscr{C}^V$ is realized by a sequence of vector bundles on $M^\infty_{n,n}$ via restriction to Z^∞ and $\underline{\pi}^c_*$. □

4.2. Isotypical ideals

Let V_λ be an irreducible rational GL_n-module with a highest weight λ. Let χ be the standard representation of $GL_1(=\mathbb{G}_m)$. For each $\gamma \in X^*(T_n)$ and a $\mathfrak{gl}_n$-module W, we denote by $W_{[\gamma]}$ the γ-isotypical component of W. For each $i = 1, \ldots, n$, we put

$$S^p_{\langle i\rangle}(V^*) := \sum_{\lambda \in \mathrm{wt} S^p(V^*)} S^p(V^*)_{[\lambda]} \otimes z_i^{\langle \omega_i^\vee, \lambda\rangle} k[Z] \subset S^p_{T_n}(V^*).$$

For each $p \in \mathbb{Z}_{\geqslant 0}$, $\lambda \in \mathrm{wt} S^p(V^*)$ and $\mu \in X^*(T_n)$, we put

$$M(p, \lambda, \mu) := z^\mu \left(S^p(V^*) \otimes k[Z] \cap \bigcap_{i=1}^n z_i^{-\langle \omega_i^\vee, \lambda\rangle} S^p_{\langle i\rangle}(V^*) \right).$$

Let θ be a T_n-weight such that $\mathrm{wt} V \subset \theta - R^+$. Then, we put

$$I(p, \lambda, \mu)^\theta := \bigoplus_{l \geqslant 0} M(p + l, \lambda - l\theta, \mu) \subset S^p_Z(V^*).$$

Lemma 4.7. *Under the above settings, $I(p, \lambda, \mu)^\theta$ is a $\mathfrak{p}^0$-stable integrally closed ideal of $k[V \times Z]$.*

Proof. Since each $S^p_{\langle i\rangle}(V^*)$ is $\mathfrak{p}^0$-stable, each $M(p, \lambda, \mu)$ is $\mathfrak{p}^0$-stable. Hence, $I(p, \lambda, \mu)^\theta$ is also $\mathfrak{p}^0$-stable. We have $(V^* \otimes k[Z]) \cdot M(p, \lambda, \mu) \subset M(p+1, \lambda - \theta, \mu)$. Thus, $I(p, \lambda, \mu)^\theta$ is an ideal of $k[V \times Z]$. Put

$$I^\theta_{\langle i\rangle} := \bigoplus_{q \geqslant 0} z_i^{q\langle \omega_i^\vee, \theta\rangle} S^q_{\langle i\rangle}(V^*).$$

Then, we have

$$I(p, \lambda, \mu)^\theta := \bigoplus_{q \geqslant p} z^\mu S^q(V^*) \otimes k[Z] \cap \bigcap_{i=1}^n z^{\mu - \langle \omega_i^\vee, \lambda + p\theta\rangle \alpha_i} I^\theta_{\langle i\rangle}.$$

Here, each $I^\theta_{\langle i\rangle}$ is a integrally closed $k[V \times Z]$-submodules in $k[V \times T_n]$. Hence, their twist by characters and their intersections are again integrally closed. $\square$

Consider a $H\,(= \mathbb{G}_m \times GL_n \times GL_n)$ -submodule

$$\chi^p \boxtimes V_{\lambda+\mu} \boxtimes V_\mu^* \subset \Gamma(M_V, F(I(p, \lambda, \mu)^\theta)) \subset k[M_V].$$

Then, we have a corresponding inclusion $\chi^p \boxtimes V_{\lambda+\mu} \boxtimes V_\mu \subset \Gamma(M^t_{n,n}, S^p_{M^t_{n,n}}(V^*))$. By composing with the restriction to Z, we have a morphism

$$\mathrm{rest} : \Gamma(M^t_{n,n}, S^p_{M^t_{n,n}}(V^*)) \to S^p(V^*) \otimes k[Z].$$

Lemma 4.8. *Under the above settings, we have*

$$\mathrm{rest}(\chi^p \boxtimes V_{\lambda+\mu} \boxtimes V_\mu^*) = \lambda \otimes z^\mu \subset S^p(V^*)_{[\lambda]} \otimes k[Z]$$

as a T_n-submodule.

Proof. Let $\mathscr{L}_\mu$ be a $GL_n \times GL_n$-equivariant line bundle on $M^t_{n,n}$ generated by $V_\mu \boxtimes V^*_\mu \subset k[M_{n,n}]$. Then, we have $\chi^p \boxtimes V_{\lambda+\mu} \boxtimes V^*_\mu \subset \Gamma(M^t_{n,n}, S^p(V^*) \otimes \mathscr{L}_\mu)$. Let z_0 be a unique T_n-fixed point of Z. Since rest is equivalent to take the fiber of $S^p(V^*) \otimes \mathscr{L}_\mu$ along z_0, we have

$$\mathrm{rest}(S^0(V^*) \otimes (V_\mu \boxtimes V^*_\mu)) = k \otimes z^\mu.$$

Thus, we deduce

$$\mathrm{rest}(S^p(V^*) \otimes (V_\mu \boxtimes V^*_\mu)) = S^p(V^*) \otimes z^\mu.$$

Here Hz_0 is a product of two flag varieties of GL_n. Therefore, we have

$$\mathrm{rest}(\chi^p \boxtimes V_{\lambda+\mu} \boxtimes V^*_\mu) \neq 0.$$

Hence, we obtain the result by the comparison of weights. □

We define $L(p, \lambda, \mu)$ to be the χ^p-isotypical component of $\Gamma(M_V, F(I(p, \lambda, \mu)^\theta))$. We call $L(p, \lambda, \mu)$ quasi-spanned if $\dim \mathrm{Hom}_H(\chi^p \boxtimes V_{\lambda+\mu} \boxtimes V^*_\mu, k[M_V]) \geqslant 1$.

Lemma 4.9. *$L(p+q, \lambda+\lambda', \mu+\mu')$ is quasi-spanned if both $L(p, \lambda, \mu)$ and $L(q, \lambda', \mu')$ are quasi-spanned.*

Proof. Since $k[M_V]$ is integral, the multiplication of highest weight vectors in $(\chi^p \boxtimes V_{\lambda+\mu} \boxtimes V^*_\mu) \cdot (\chi^q \boxtimes V_{\lambda'+\mu'} \boxtimes V^*_{\mu'})$ is non-zero. Hence the result follows. □

4.3. Spherical cases and Young diagrams

We retain the setting of the previous subsection. We restrict ourselves to the case that V is an irreducible $\mathfrak{gl}_n$-module of dimension n. In this case, $V \oplus M_{n,m}$ is a spherical H-variety. Therefore, we can express everything explicitly.

Corollary 4.10. *Assume that V is a n-dimensional irreducible GL_n-module with a highest weight ω. If $L(p, \lambda, \mu)$ is quasi-spanned, then $F(I(p, \lambda, \mu)^\omega) \subset k[M_V]$ is the minimal H-invariant integrally closed ideal which contains $\chi^p \boxtimes V_{\lambda+\mu} \boxtimes V_\mu$.*

Proof. By assumption, $S^p(V^*)$ is an irreducible $\mathfrak{gl}_n$-module which is multiplicity-free as a T_n-module. Hence, the minimal $\mathfrak{p}^0$-module in $S^p(V^*) \otimes k[Z]$ generated by $\lambda \otimes t^\mu$ is $M(p, \lambda, \mu)$. We have $V^* \otimes k[Z] \cdot M(p, \lambda, \mu) \subset M(p+1, \lambda-\omega, \mu)$ by weight computation. Moreover, we have

$$M(p+1, \lambda-\omega, \mu) \subset V^* \otimes k[Z] \cdot M(p, \lambda, \mu)$$

since the RHS contains a generator of the LHS as a $\left(\mathfrak{p}^0, k[Z]\right)$-module. Hence, Lemma 4.7 and Theorem 4.6 yields the result. □

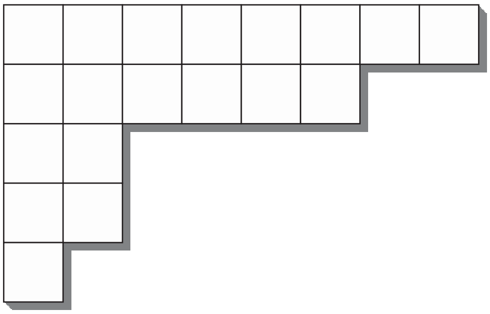

Fig. 1. $(\lambda^{(1)}, \lambda^{(2)}, \ldots) = (5, 4, 2, 2, 2, 2, 1, 1, 0, \ldots)$.

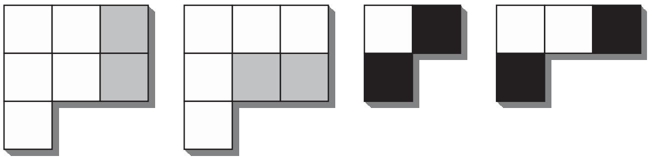

Fig. 2. Gray boxes are adjacent while black boxes are not.

Corollary 4.11. *Assume that V is a n-dimensional irreducible GL_n-module with a highest weight ω, then we have*

$$L(p, \lambda, \mu) = \left(\bigoplus_{\alpha, \beta \in R^+} V_{\lambda+\mu-\alpha} \boxtimes V^*_{\mu-\beta} \right) \cap S^p(V^*) \otimes k[M_{n,n}].$$

Proof. $S^p(V^*)$ is an irreducible GL_n-module. Thus, we have

$$L(p, \lambda, \mu) = \sum_{\alpha, \beta \in R^+} L(p, \lambda - \alpha + \beta, \mu - \beta)$$

by a reformulation of the $\mathfrak{p}^0$-action and $k[Z]$-action. Therefore, the formula follows since $S^p(V^*) \otimes k[M_{n,n}]$ is multiplicity-free as a H-module. □

Definition 4.12 (*Root and weight orderings*). Let $\lambda, \mu \in X^*(T_n)$ be two weights. Then, we write $\lambda \geqslant \mu$ if and only if $\lambda \in \mu + \sum_{i=2}^n \mathbb{Z}_{\geqslant 0} \alpha_i$. Similarly, we write $\lambda \supset \mu$ if and only if $\lambda \in \mu + \sum_{i=1}^n \mathbb{Z}_{\geqslant 0} \omega_i$. We write $\lambda = \sum_{i=1}^n \lambda^{(i)} \omega_i$. Then, we define $|\lambda| = \sum_{i=1}^n \lambda^{(i)}$. We call λ dominant if and only if $\lambda^{(1)} \geqslant \lambda^{(2)} \geqslant \cdots \geqslant \lambda^{(n)}$ holds.

Let $\lambda = \sum_{i=1}^n \lambda^{(i)} \omega_i$ be a T_n-weight such that $\lambda \supset 0$. Then, we identify λ with a Young diagram as in Fig. 1.

We define the degree of λ as $\deg \lambda = \sum_{i=1}^n \lambda^{(i)}$. We call a pair of boxes in a Young diagram adjacent if they share one edge (Fig. 2).

We put $\bar{\omega}_q := \sum_{j=1}^{q} \omega_j$ for $q = 1, \ldots, n$ and $\bar{\omega}_0 = 0$. We have $\deg \bar{\omega}_q = q$. Here $\omega_1 = \bar{\omega}_1$ and $\bar{\omega}_{n-1}$ corresponds to n-dimensional irreducible GL_n-modules.

Example 4.13. Assume that $V^* = V_{\omega_1}$. For each $1 \leqslant j \leqslant n$, $L(0, 0, \bar{\omega}_j) \subset k[M_{n,n}]$ is an integrally closed ideal which contains $V_{\bar{\omega}_j} \boxtimes V^*_{\bar{\omega}_j}$ as a unique H-submodule with a Young diagram of degree (j, j). Similarly, for each $1 \leqslant j \leqslant n-1$, $L(1, \omega_{j+1}, \bar{\omega}_j) \subset V^* \otimes k[M_{n,n}]$ is a $(H, k[M_{n,n}])$-submodule which contains $V_{\bar{\omega}_{j+1}} \boxtimes V^*_{\bar{\omega}_j}$ as a unique H-submodule with a Young diagram of degree $(j+1, j)$.

4.4. Brion's problem

We retain the settings of the previous subsection. In particular, $X_V = V \oplus M_{n,m}$ is a spherical H-variety. For an ideal J, we denote its integral closure by $\bar{J}$. In [4], Brion has described integrally closed invariant ideals on an affine spherical variety. There he raised the following question.

Problem 4.14. Let I be a <u>prime</u> integrally closed invariant ideal of an affine spherical variety. Then, is it true that $\overline{(I^m)} = (\bar{I})^m$ for every positive integer m?

To provide a positive answer to Brion's problem for our case, we need the following:

Theorem 4.15. *Let $L(p, \lambda_1, \mu_1), L(q, \lambda_2, \mu_2)$ be two quasi-spanned submodules of $k[M_V]$. Then, the multiplication map*

$$L(p, \lambda_1, \mu_1) \otimes L(q, \lambda_2, \mu_2) \to L(p+q, \lambda_1 + \lambda_2, \mu_1 + \mu_2)$$

is surjective.

We postpone the proof of Theorem 4.15 until Section 5.1. The validity of Brion's Problem (in our case) is a direct consequence of Theorem 4.15:

Corollary 4.16. *Brion's problem is true for $V \oplus M_{n,m}$, where V is an irreducible n-dimensional GL_n-module.*

For simplicity, we denote $S^p(V^*)$ by S^p and denote $\wedge^q V^*$ by $\wedge^q$. The structure of the multiplicity-free M_V is given as follows:

Theorem 4.17 (*Leahy [18], Benson and Ratcliff [1]*). *The set of $T_n \times T_m$-highest weight vectors of $k[V \oplus M_{n,m}]$ forms a free monoid in $X^*(T_n \times T_m)$ generated by the following sets*:

Case: $V^* = V_{\omega_1}$: $(\bar{\omega}_1, -\bar{\omega}_1), \ldots, (\bar{\omega}_n, -\bar{\omega}_n)$ *and* $(\bar{\omega}_1, 0), (\bar{\omega}_2, -\bar{\omega}_1), \ldots, (\bar{\omega}_n, -\bar{\omega}_{n-1})$;
Case: $V^* = V_{\bar{\omega}_n}$: $(\bar{\omega}_1, -\bar{\omega}_1), \ldots, (\bar{\omega}_n, -\bar{\omega}_n)$ *and* $(0, -\bar{\omega}_1), (\bar{\omega}_1, -\bar{\omega}_2), \ldots, (\bar{\omega}_{n-1}, -\bar{\omega}_n)$.
In both cases, corresponding H-representations sit inside of $k[M_{n,m}] \oplus V^* \otimes k[M_{n,m}]$.

From now on, we assume that $V^* = V_{\omega_1} = V_{\bar{\omega}_1}$ for the sake of simplicity.

Proof of Corollary 4.16. By Corollary 4.10, every minimal H-invariant integrally closed ideal of $k[V \oplus M_{n,n}]$ is of the form $\bigoplus_{q \geqslant 0} L(p+q, \lambda - q\omega_1, \mu)$. By Theorem 4.15, we know that any multiplication of such kind of ideals are again minimal. Thus, the resulting ideal is again integrally closed by Lemma 4.9. By Corollary 4.11, we have $L(0, 0, \bar{\omega}_{i+1}) \subset L(0, 0, \bar{\omega}_i)$ and $L(1, \omega_{i+1}, \bar{\omega}_i) \subset L(1, \omega_i, \bar{\omega}_{i-1})$ for every $i = 1, \dots, n$. By Theorem 4.17 and 4.15, a prime ideal of $V \oplus M_{n,n}$ is a minimal H-invariant integrally closed ideal or a sum two minimal H-invariant integrally closed ideal of $k[V \oplus M_{n,n}]$ generated by $L(0, 0, \bar{\omega}_r)$ and $L(1, \omega_s, \bar{\omega}_{s-1})$(here $r > s$ are integers). Since the assertion is true for the former case, we assume the latter setting in the below.

Here, we have $L(1, \omega_1, \bar{\omega}_r) \subset L(1, \omega_s, \bar{\omega}_{s-1})$. As a consequence, the m-th power I^m of the H-invariant ideal, which is generated by $(L(0, 0, \bar{\omega}_r) + L(1, \omega_s, \bar{\omega}_{s-1}))^m$, has $L(q, q\omega_s, q\bar{\omega}_{s-1} + (m-q)\bar{\omega}_r)$ as its degree q-part $I^m(q)$ for $q = 0, \dots, m$. Hence, if we have $q_1 + q_2 = q$, then we have $I^m(q_1) \cdot I^m(q_2) \subset I^m(q)$. It follows that the resulting ideal is integrally closed if the ideal generated by $I^m(q)$ is integrally closed. Therefore, Corollary 4.10 yields the result. $\square$

The proof of Theorem 4.15 requires a sequence of Lemmas about representation theory. We prove such results in Section 5.2 (only for the case that V^* is the standard representation of GL_n).

Lemma 4.18. *Assume that the following GL_n-module morphisms are nontrivial.*

$$S^p \otimes V_\mu \otimes \wedge^q \to S^p \otimes V_{\mu+\gamma} \to V_{\lambda+\mu+\gamma}.$$

Here the first morphism is the tensor product of the identity map of S^p and a nonzero map $V_\mu \otimes \wedge^q \to V_{\mu+\gamma}$. Then, there exists a T_n-weight $\lambda_+ \geqslant \lambda$ with the following properties:

- $0 \neq V_{\lambda_+ + \mu} \subset S^p \otimes V_\mu$;
- $0 \neq V_{\lambda+\mu+\gamma} \subset V_{\lambda_+ + \mu} \otimes \wedge^q$;
- λ_+ *is minimal among the weights with the above two properties with respect to* $\geqslant$.

Theorem 4.19. *Under the same settings as in Lemma* 4.18, *there exists a weight* $\lambda' \leqslant \lambda_+$ *such that*:

- $0 \neq V_{\lambda'+\mu} \subset S^p \otimes V_\mu$;
- $0 \neq V_{\lambda+\mu+\gamma} \subset V_{\lambda'+\mu} \otimes \wedge^q$;
- *The composition map*

$$V_{\lambda+\mu+\gamma} \hookrightarrow V_{\lambda'+\mu} \otimes \wedge^q \hookrightarrow S^p \otimes V_\mu \otimes \wedge^q \to S^p \otimes V_{\mu+\gamma} \to V_{\lambda+\mu+\gamma}.$$

  is an isomorphism.

Proposition 4.20. *Let μ be a Young diagram with p-rows. Let $\mathscr{R}$ denote the following composition map of nontrivial GL_n-modules*:

$$\mathscr{R} : S^p \otimes V_\mu \otimes V^* \otimes \wedge^q \to S^{p+1} \otimes V_\mu \otimes \wedge^q \to S^{p+1} \otimes V_{\mu+\gamma} \to V_{\lambda+\mu+\gamma},$$

where the first map is a contraction $S^p \otimes V^* \to S^{p+1}$ of the first and the third factors, the second map is some projection $V_\mu \otimes \wedge^q \to V_{\mu+\gamma}$, and the third map is also some projection. Here we assume that $(\lambda+\mu+\gamma)$ is a Young diagram with $(p+1)$-rows. Let $\lambda' \in \mathrm{wt} S^p$ be such that (1) *$(\lambda'+\mu)$ is a Young diagram with p-rows and* (2) *there exists a sequence of inclusions $\mathscr{L} : V_{\lambda+\mu+\gamma} \subset V_{\lambda'+\mu} \otimes \wedge^{q+1} \subset S^p \otimes V_\mu \otimes \wedge^{q+1} \subset S^p \otimes V_\mu \otimes V^* \otimes \wedge^q$. Then, the composition map $\mathscr{R} \circ \mathscr{L}$ is an isomorphism.*

5. Proof of Theorems

5.1. Proof of Theorem 4.15

We prove only the case $V^* = V_{\omega_1} = \wedge^1$. The other case is similar by taking lowest weight vectors instead of highest weight vectors. By Theorem 4.17 and the associativity of multiplication map, all we have to do is to prove the surjectivity result for an arbitrary quasi-spanned $L(p,\lambda,\mu)$ and (**a**) each of $L(0,0,\bar{\omega}_1), \ldots, L(0,0,\bar{\omega}_n)$ or (**b**) each of $L(1,\omega_1,0), L(1,\omega_2,\bar{\omega}_1), \ldots, L(1,\omega_n,\bar{\omega}_{n-1})$.

First, we prove the case (**a**). We have $\wedge^i \boxtimes (\wedge^i)^* \subset L(0,0,\bar{\omega}_i) \subset k[M_{n,n}]$. Let $V_\mu \otimes \wedge^i \to V_{\mu+\gamma}$ be an arbitrary GL_n-module projection. Then, the image of the multiplication map $V_\mu \boxtimes V_\mu^* \cdot \wedge^i \boxtimes (\wedge^i)^*$ in $k[M_{n,n}]$ contains $V_{\mu+\gamma} \boxtimes V_{\mu+\gamma}^*$ (cf. [8] or [21, Theorem 3.7]).

Claim 2. *Let $V_{\lambda^0+\mu+\gamma} \boxtimes V^*_{\mu+\gamma} \subset L(p,\lambda,\mu+\bar{\omega}_i)$. Then, we have $V_{\lambda^0+\mu+\gamma} \boxtimes V^*_{\mu+\gamma} \subset L(p,\lambda,\mu) \cdot L(0,0,\bar{\omega}_i)$.*

Proof. Let $\lambda'+\mu' \leqslant \lambda+\mu$ and $\mu' \leqslant \mu$ be weights such that both $\lambda'+\mu'$ and μ' are dominant. We can assume

$$V_{\lambda^0+\mu+\gamma} \boxtimes V^*_{\mu+\gamma} \not\subset L(p,\lambda',\mu'+\bar{\omega}_i) \tag{5.1}$$

for every possible $(\lambda',\mu') \neq (\lambda,\mu)$. Let $\gamma^1 \in \mathrm{wt}\wedge^i$ be the maximal weight such that $\mu^1 := \mu+\gamma-\gamma^1$ is dominant. We have $\mu^1 \leqslant \mu$. Let λ^1_+ be the weight corresponding to λ_+ in Lemma 4.18 with respect to $(\lambda,\mu,\gamma) = (\lambda^0,\mu^1,\gamma^1)$. The weight $(\mu+\gamma)$ is made from $(\mu+\bar{\omega}_i)$ by subtracting $\{\omega_j\}_{j\leqslant i}$ and adding $\{\omega_j\}_{j>i}$ (at most once for each). Let R_1 be the lowest row in $(\lambda+\mu+\bar{\omega}_i)$ such that $R_1 \cap (\lambda+\mu+\bar{\omega}_i) = \bar{\omega}_i$. Let $R_2 \neq R_1$ be (1) the highest row such that $R_1 \cap (\lambda+\mu+\bar{\omega}_i) = \bar{\omega}_i$ or (2) the up-adjacent row of R_1. Let Ω_i be the column of $(\lambda^0+\mu+\gamma)$ which corresponds to the weight ω_i. Let $\hat{\lambda} := (\lambda+\mu+\bar{\omega}_i)\backslash(\mu+\bar{\omega}_i) \subset (\lambda+\mu+\bar{\omega}_i)$ and $\hat{\lambda}^0 := (\lambda^0+\mu+\gamma)\backslash(\mu+\gamma) \subset (\lambda^0+\mu+\gamma)$ be subsets of Young diagrams. If we have $(\hat{\lambda}^0 \cap R_1) \neq \emptyset$, then it corresponds to some box of λ which sits left or right to Ω_i by (5.1). If $(\hat{\lambda}^0 \cap R_1) = \emptyset$ or $(\hat{\lambda}^0 \cap R_1)$ comes from the left of Ω_1, we conclude $\lambda^1_+ + \mu^1 \leqslant \lambda+\mu$ by (5.1). Therefore, we have

$$V_{\lambda^1_+ + \mu^1} \boxtimes V_{\mu^1} \subset L(p,\lambda,\mu^0).$$

By Theorem 4.19, we have a surjection

$$V_{\lambda^0+\mu+\gamma} = V_{\lambda^0+\mu^1+\gamma^1} \hookrightarrow \bigoplus_{\beta \leqslant 0} V_{\lambda^1_+ + \mu^1 + \beta} \otimes \wedge^i \hookrightarrow S^p(V^*) \otimes V_{\mu^1} \otimes \wedge^i \to S^p(V) \otimes V_{\mu+\gamma} \to V_{\lambda^0+\mu+\gamma}.$$

Together with the natural surjection

$$V_{\mu+\gamma} \hookrightarrow V_{\mu^1} \otimes \wedge^i \to V_{\mu+\gamma},$$

we obtain

$$V_{\lambda^0+\mu+\gamma} \boxtimes V_{\mu+\gamma} \subset (V_{\lambda^1_+ + \mu^1} \boxtimes V_{\mu^1}) \cdot (\wedge^i \boxtimes (\wedge^i)^*) \subset L(p, \lambda, \mu) \cdot L(0, 0, \bar{\omega}_i) \subset L(p, \lambda, \mu + \bar{\omega}_i)$$

as desired.

Therefore, we assume $(\hat{\lambda}^0 \cap R_1)$ comes from the right of Ω_i in the below.

For every $V_\xi \boxtimes V^*_\chi, V_{\xi'} \boxtimes V^*_{\chi'} \subset k[M_V]$, we have $V_{\xi+\xi'} \boxtimes V^*_{\chi+\chi'} \subset k[M_V]$. By the associativity of the multiplication, the assertion for $V_{\xi+\xi'} \boxtimes V^*_{\chi+\chi'}$ follows from the corresponding assertion for $V_\xi \boxtimes V^*_\chi$.

Let R_j be the row such that $\hat{\lambda} \cap R_j \neq \emptyset$ and $\hat{\lambda}^0 \cap R_j = \emptyset$. According to $h = |\hat{\lambda}^0 \cap (R_1 \cup R_2 \cup R_j)|$, there are three cases:

Case $h = 0$: This case reduces to the case of [8] if we choose $\xi = \chi = (\lambda^0 + \mu + \gamma) \cap (R_1 \cup R_2) = (\mu + \gamma) \cap (R_1 \cup R_2)$;

Case $h = 1$: By choosing $\xi = (\lambda^0 + \mu + \gamma) \cap (R_1 \cup R_2)$ and $\chi = (\mu + \gamma) \cap (R_1 \cup R_2)$, the assertion reduces to

$$V_{\bar{\omega}_{s+1}+\bar{\omega}_{t+i-s}} \boxtimes V_{\bar{\omega}_s+\bar{\omega}_{t+i-s}} \subset L(1, \omega_{t+1}, \bar{\omega}_t) \cdot L(0, 0, \bar{\omega}_i),$$

where $s \leqslant t$;

Case $h = 2$: By choosing $\xi = (\lambda^0 + \mu + \gamma) \cap (R_1 \cup R_2 \cup R_j)$ and $\chi = (\mu + \gamma) \cap (R_1 \cup R_2 \cup R_j)$, the assertion reduces to

$$V_{\bar{\omega}_{s+1}+\bar{\omega}_{t+1}+\bar{\omega}_{j+1}} \boxtimes V_{\bar{\omega}_s+\bar{\omega}_t+\bar{\omega}_{j+1}} \subset L(1, \omega_{t'+1}, \omega_{t'}) \cdot L(1, \omega_{j+1}, \bar{\omega}_j) \cdot L(0, 0, \bar{\omega}_i),$$

where $s \leqslant t' \leqslant t < j$.

The case $h = 1$ follow from the proof of Theorem 4.19 (${}^L w \neq {}^R w$ for corresponding weights). The case $h = 2$ is a consequence of the case $h = 1$, Proposition 4.20, and the associativity. □

Here, the rest of H-submodules of $L\,(p, \lambda, \mu + \bar{\omega}_i)$ are contained in

$$\sum_{j \neq i} L(p, \lambda + \alpha_j, \mu - \alpha_j + \bar{\omega}_i) + \sum_{j \neq i} L(p, \lambda, \mu - \alpha_j + \bar{\omega}_i)$$

since $\bar{\omega}_i - \alpha_i \in \mathrm{wt} \wedge^i$. Here $\mu - \alpha_j$ is dominant if both μ and $\mu - \alpha_j + \bar{\omega}_i$ are dominant. Therefore, a downward induction yields case (**a**).

Next, we prove case (**b**). By the associativity of multiplication, we have only to prove surjectivity for $L(p, \lambda, \mu)$ which is written as a product of $L(1, \omega_1, 0), L(1, \omega_2, \bar{\omega}_1), \ldots, L(1, \omega_n, \bar{\omega}_{n-1})$. We have $\wedge^i \boxtimes (\wedge^{i-1})^* \subset L(1, \omega_i, \bar{\omega}_{i-1})$. Let $V_{\lambda+\mu} \boxtimes V_\mu^* \subset L(p, \lambda, \mu)$. Let $V_\mu \otimes \wedge^i \to V_{\mu+\gamma'}$ and $V_\mu \otimes \wedge^{i-1} \to V_{\mu+\gamma}$ be arbitrary GL_n-module projections. Let $\lambda^0 \leqslant \lambda$ be a weight such that $\lambda^0 + \mu$ is dominant. Then, Proposition 4.20 yields that every H-submodule $V_{\lambda^0+\mu+\gamma'} \boxtimes V^*_{\mu+\gamma} \subset L(p+1, \lambda + \omega_i, \mu + \bar{\omega}_{i-1})$ is in the image of the multiplication map $L(p, \lambda, \mu) \cdot L(1, \omega_i, \bar{\omega}_{i-1})$. Therefore, we can reduce the surjectivity problem to

$$\sum_{j \neq i} L(p+1, \lambda + \omega_i + \alpha_j, \mu - \alpha_j + \bar{\omega}_{i-1}) + \sum_{j \neq i} L(p+1, \lambda + \omega_i, \mu - \alpha_j + \bar{\omega}_{i-1})$$

as in the proof of case (**a**). Thus, a downward induction also yields the result.

5.2. *Proofs of representation-theoretic results*

Lemma 5.1 (*See e.g. Fulton and Harris [12, (6.8) and (6.9)]*). *We have the following two formulas*:

(1) $V_\mu \otimes \wedge^q$ *is a direct sum of all* V_{μ_+}*'s such that* (1) μ_+ *is obtained by adding q-boxes to* μ, *and* (2) *no two added boxes are in the same column*;
(2) $S^p \otimes V_\mu$ *is a direct sum of all* V_{μ_+}*'s such that* (1) μ_+ *is obtained by adding p-boxes to* μ, *and* (2) *no two added boxes are in the same row*.

Proof of Lemma 4.18. We have $\mu \subset (\mu + \gamma) \subset (\lambda + \mu + \gamma)$. We label (1) boxes in μ by white, (2) boxes in $(\mu + \gamma) \backslash \mu$ by black, and (3) boxes in $(\lambda + \mu + \gamma) \backslash (\mu + \gamma)$ by gray. We put $\mu^R := \mu + \bar{\omega}_n$. Then, we have $(\mu + \gamma) \subset \mu^R$ by Lemma 5.1(1). We put $\mu^{LR} := (\mu^{(1)} + p + 1)\omega_1 + \sum_{i=1}^{n-1} (\mu^{(i)} + 1)\omega_{i+1}$ and $\mu^L := (\mu^{(1)} + p)\omega_1 + \sum_{i=1}^{n-1} \mu^{(i)} \omega_{i+1}$. By Lemma 5.1(2), we have $(\lambda + \mu + \gamma) \subset \mu^{LR}$. Here, every boxes in $(\lambda + \mu + \gamma)$, which is outside of $\mu^R \cup \mu^L$, is labeled by gray. We exchange gray and black boxes by the following procedure (cf. Fig. 3 and Fig. 4):

(1) Set the current column as the most-right column;
(2) At the current column, do the following operations:
   (a) If that column contains only grayboxes and one of its left-next box is black, swap the gray box with its (left)-adjacent black box;
   (b) If that column contains both gray and black boxes, swap the black box with the bottom (gray) box in that column;
(3) Change the current column to the left-next column and goes to back to (2) if the left-next column exists.

It is obvious that this procedure terminates and yields another Young diagram (formed by white and gray boxes) $(\lambda_+ + \mu)$ with $|\lambda_+| = |\lambda| (= p)$. We prove that λ_+ satisfies the desired property. Since each procedure changes λ by adding a positive root (case 2(b)) or do

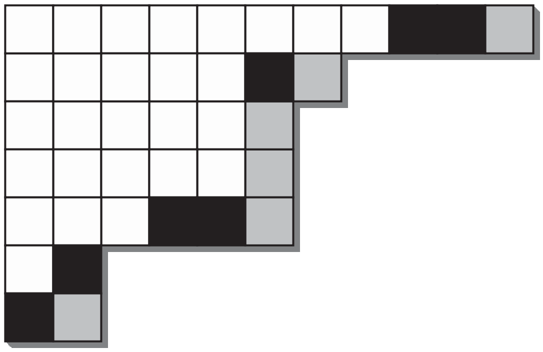

Fig. 3. Before the procedure.

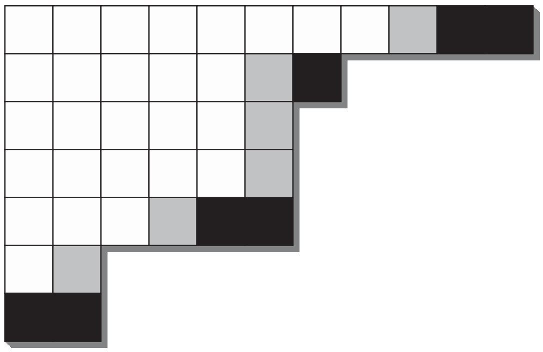

Fig. 4. After the procedure.

nothing (all other cases), we have $\lambda_+ \geqslant \lambda$. For each procedure, no gray box is right-next to a black box at the current column and no gray box is left-next to a gray box. Thus, we have $V_{\lambda_+ + \mu} \subset S^p \otimes V_\mu$. After each procedure, each column with more than one gray or black box has at most one black box at the bottom. Therefore, we have $V_{\lambda+\mu+\gamma} \subset V_{\lambda_+ + \mu} \otimes \wedge^q$. In each procedure, one adds different simple roots which is needed to make $\lambda + \mu$ into a Young diagram. Hence, the minimality assertion follows. $\square$

We denote some vector in V_μ of weight μ' by $v^\mu_{\mu'}$ (it is not unique nor exists in general). Similarly, we fix a T_n-eigenbasis of S^p (resp. $\wedge^q$) parametrized by its weight as $\{v^p_\lambda\}_{\lambda \in \mathrm{wt} S^p}$ (resp. $\{{}^q v_\lambda\}_{\lambda \in \mathrm{wt} \wedge^q}$).

Lemma 5.2. *Let $v^*_\mu \in V^*_\mu$ be a lowest weight vector. Let* $\mathrm{pr} : S^p \otimes V_\mu \otimes \wedge^q \to S^p \otimes \wedge^q$ *be a vector space contraction with respect to v^*_μ (hence, it is only a $\mathfrak{b}^-$-module map). Then, the image of highest weight vectors in $S^p \otimes V_\mu \otimes \wedge^q$ are linearly independent.*

Proof. We define a set Φ as follows:

$$\Phi := \{(\lambda, \gamma) \in \mathrm{wt} S^p \times \mathrm{wt} \wedge^q ; V_{\lambda+\mu} \subset S^p \otimes V_\mu, V_{\lambda+\mu+\gamma} \subset V_{\lambda+\mu} \otimes \wedge^q\}.$$

Then, we have

$$S^p \otimes V_\mu \otimes \wedge^q \cong \bigoplus_{(\lambda,\gamma)\in\Phi} V_{\lambda+\mu+\gamma}.$$

Let $V_{\lambda+\mu} \subset S^p \otimes V_\mu$. Then, we have

$$v^{\lambda+\mu}_{\lambda+\mu} = \sum_{\alpha \geqslant 0} v^p_{\lambda+\alpha} \otimes v^\mu_{\mu-\alpha}.$$

Here the term $v^p_\lambda \otimes v^\mu_\mu$ is nonzero since $v^{\lambda+\mu}_{\lambda+\mu}$ is a highest weight vector. Similarly, for each $(\lambda, \gamma) \in \Phi$, a highest weight vector of $V_{\lambda+\mu+\gamma} \subset V_{\lambda+\mu} \otimes \wedge^q \subset S^p \otimes V_\mu \otimes \wedge^q$ is expressed as follows:

$$v^{\lambda+\mu+\gamma}_{\lambda+\mu+\gamma} = \sum_{\alpha,\beta \geqslant 0} X_{-\beta}(v^p_{\lambda+\alpha} \otimes v^\mu_{\mu-\alpha}) \otimes^q v_{\gamma+\beta}.$$

Here $X_{-\beta} \in U(\mathfrak{g})$ is some element of weight $-\beta$. The term $\sum_{\alpha \geqslant 0} v^p_{\lambda+\alpha} \otimes v^\mu_{\mu-\alpha} \otimes^q v_\gamma$ is again non-zero. Therefore,

$$\mathrm{pr}(v^{\lambda+\mu+\gamma}_{\lambda+\mu+\gamma}) = \sum_{\beta \geqslant 0} v^p_{\lambda-\beta} \otimes^q v_{\gamma+\beta}$$

contains a nonzero term $v^p_\lambda \otimes^q v_\gamma$. Since the appearance pattern of terms are upper triangular with respect to γ, we obtain the result. □

Proof of Theorem 4.19. We define two subsets $\Phi^L, \Phi^R \subset \mathrm{wt} S^p \times \mathrm{wt} \wedge^q$ as follows:

$$\Phi^L := \{(\lambda', \gamma') \in \mathrm{wt} S^p \times \mathrm{wt} \wedge^q ; 0 \neq V_{\lambda'+\mu} \subset S^p \otimes V_\mu, V_{\lambda+\mu+\gamma} \subset V_{\lambda'+\mu} \otimes \wedge^q,$$
$$\lambda' + \gamma' = \lambda + \gamma\},$$

$$\Phi^R := \{(\lambda'', \gamma'') \in \mathrm{wt} S^p \times \mathrm{wt} \wedge^q ; 0 \neq V_{\mu+\gamma''} \subset V_\mu \otimes \wedge^q, V_{\lambda+\mu+\gamma} \subset S^p$$
$$\otimes V_{\mu+\gamma''}, \lambda'' + \gamma'' = \lambda + \gamma\}.$$

If we replace (λ, γ) by $(\lambda'', \gamma'') \in \Phi^R$, then the assumption of Lemma 4.18 holds. Hence, we have λ''_+ constructed from λ'' as in Lemma 4.18. We put $\gamma''_+ := \lambda + \gamma - \lambda''_+$. Then, we have $(\lambda''_+, \gamma''_+) \in \Phi^L$.

Lemma 5.3. *The assignment $\Phi^R \ni (\lambda'', \gamma'') \mapsto (\lambda''_+, \gamma''_+) \in \Phi^L$ is a one-to-one mapping.*

Proof. If we transpose the Young diagram by changing columns to rows, the conditions of Lemma 5.1(1) and (2) become another. In particular, the procedure in the proof of Lemma 4.18 yields the reverse correspondence of the assignment. Hence, the result follows. □

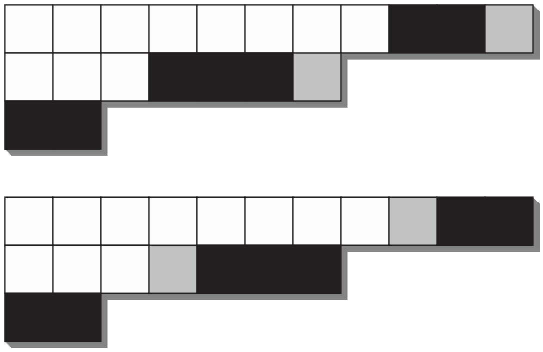

Fig. 5. After we cut-off some gray and white rows from Fig. 3 and 4.

Lemma 5.4. *Let $(\lambda^0, \gamma^0), (\lambda^1, \gamma^1) \in \Phi^R$ be two elements. Then, we have $\lambda^0 \geqslant \lambda^1$ if and only if $\lambda^0_+ \geqslant \lambda^1_+$. Moreover, we have $\lambda^0_+ \geqslant \lambda^1$ only if $\lambda^0_+ \geqslant \lambda^1_+$.*

Proof. We adapt the same notations as in the proof of Lemma 4.18. Here μ is common for all. In particular, white boxes of the proof of Lemma 4.18 is fixed for every choice of $(\lambda, \gamma) \in \Psi^R \cup \Psi^L$. Hence, if a row in $(\lambda + \mu + \gamma)$ does not contain gray or black box, then this row never contribute the difference among λ's. Therefore, we can delete all such rows in order to simplify the situation. By Lemma 5.1, every non-white boxes in one column is labeled by gray except for one (possible) black box. Hence, such a row never contribute the difference among λ's. Therefore, the difference among λ's are unchanged if we swap the black box in a column to the bottom and cut-off the rectangular region from the Young diagram which have gray wall in one column and white boxes at its left (cf. Fig. 5).

Then, each row has at most one gray box and the assignment $\lambda \mapsto \lambda_+$ moves the gray box from the most right column to the most left column within the intersection of the row and $(\lambda + \mu + \gamma)\backslash\mu$. Therefore, the result follows. $\square$

For each $(\lambda', \gamma') \in \Phi^L$, we denote a highest weight vector of $V_{\lambda+\mu+\gamma} \subset V_{\lambda'+\mu} \otimes \wedge^q \subset S^p \otimes V_\mu \otimes \wedge^q$ by ${}^L w_{\lambda',\gamma'}$. For each $(\lambda'', \gamma'') \in \Phi^R$, we denote a highest weight vector of $V_{\lambda+\mu+\gamma} \subset S^p \otimes V_{\mu+\gamma''} \subset S^p \otimes V_\mu \otimes \wedge^q$ by ${}^R w_{\lambda'',\gamma''}$. By the same calculation as in the proof of Lemma 5.2, we have

$$\mathrm{pr}({}^L w_{\lambda',\gamma'}) = \sum_{(\lambda,\gamma)\in\Phi^L} c^{\lambda}_{\lambda'} v^p_\lambda \otimes^q v_\gamma$$

$$\mathrm{pr}({}^R w_{\lambda'',\gamma''}) = \sum_{(\lambda,\gamma)\in\Phi^R} d^{\lambda}_{\lambda''} v^p_\lambda \otimes^q v_\gamma$$

Here $\{c^{\lambda}_{\lambda'}\}$ is a lower triangular matrix and $\{d^{\lambda}_{\lambda''}\}$ is an upper triangular matrix with respect to $\geqslant$. By the proof of Lemma 5.2, all diagonal entry of the both matrices are nonzero. For each $(\lambda^0, \gamma^0) \in \Phi^R$, $\{\mathrm{pr}({}^R w_{\lambda'',\gamma''})\}_{\lambda'' \not\leqslant \lambda^0} \cup \{\mathrm{pr}({}^L w_{\lambda',\gamma'})\}_{\lambda' \leqslant \lambda^0_+}$ is a set of linearly independent vectors. Hence, we obtain the result. $\square$

Proof of Proposition 4.20. Since $\lambda+\mu+\gamma$ is a Young diagram with $(p+1)$-rows, it follows that $V_{\lambda+\mu+\gamma} \subset S^{p+1} \otimes V_\mu \otimes \wedge^q$ is multiplicity free. Hence, all we have to prove is that the image of $\mathscr{L}$ is nonzero under the contraction map $S^p \otimes V_\mu \otimes V^* \otimes \wedge^q \to S^{p+1} \otimes V_\mu \otimes \wedge^q$. By the same calculation as in the proof of Lemma 5.2, we know that a highest weight vector of $V_{\lambda+\mu+\gamma} \subset V_{\lambda'+\mu} \otimes \wedge^{q+1} \subset S^p \otimes V_\mu \otimes \wedge^{q+1}$ is expressed as follows:

$$v^{\lambda+\mu+\gamma}_{\lambda+\mu+\gamma} = \sum_{\alpha,\beta \geqslant 0} X_{-\beta}(v^p_{\lambda'+\alpha} \otimes v^\mu_{\mu-\alpha}) \otimes^{q+1} v_{\lambda+\gamma-\lambda'+\beta}.$$

Here $X_{-\beta} \in U(\mathfrak{gl}_n)$ is some element of weight $-\beta$. The term $v^p_{p\omega_1} \otimes v^\mu_{\mu+\lambda'-p\omega_1} \otimes^{q+1} v_{\lambda+\gamma-\lambda'}$ in the above expression is non-zero. $\lambda' + \mu$ is a Young diagram with p-rows, while $\lambda+\mu+\gamma$ has $(p+1)$-rows. Hence, we have $\lambda+\gamma-\lambda' \supset \omega_1$. It follows that the image of $v^p_{p\omega_1} \otimes v^\mu_{\mu+\lambda'-p\omega_1} \otimes^{q+1} v_{\lambda+\gamma-\lambda'}$ by the sequence of morphisms

$$S^p \otimes V_\mu \otimes \wedge^{q+1} \hookrightarrow S^p \otimes V_\mu \otimes V^* \otimes \wedge^q \to S^{p+1} \otimes V_\mu \otimes \wedge^q \to V_\mu \otimes \wedge^q$$

is non-zero, where the third map is a contraction with a lowest weight vector of $S^p(V) = (S^p)^*$. Thus, the result follows. □

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